

THE KRONECKER-GAUSSIAN CURVATURE OF HYPERSPACE*

BY

HEINRICH MASCHKE

§ 1. *Definition of the Kronecker-Gaussian curvature.*

Let $x', x^2, \dots, x^{n+1}$ be the coördinates of an euclidean space of $n + 1$ dimensions, S_{n+1} , i. e., a space whose arc-element is determined by

$$(1) \quad ds^2 = \sum_{\lambda=1}^{n+1} (dx^\lambda)^2;$$

then we define any hypersurface, or, as we shall simply say, any space, of n dimensions, R_n , contained in S_{n+1} , by expressing each of the coördinates x^λ in terms of n independent variables $u_1, u_2, \dots, u_n$.

The arc-element of R_n is given by

$$(2) \quad ds^2 = \sum_{i,k=1}^n a_{ik} du_i du_k,$$

where

$$(3) \quad a_{ik} = \sum_{\lambda=1}^{n+1} \frac{\partial x^\lambda}{\partial u_i} \frac{\partial x^\lambda}{\partial u_k},$$

or

$$(4) \quad a_{ik} = \sum_{\lambda=1}^{n+1} x_i^\lambda x_k^\lambda,$$

if we agree to indicate differentiation with respect to u_i by the lower index i .

It will be sometimes convenient to write simply

$$\sum(x) \quad \text{instead of} \quad \sum_{\lambda=1}^{n+1} (x^\lambda),$$

where (x^λ) stands for any quantity involving or defined by x^λ , e. g.,

$$(5) \quad a_{ik} = \sum x_i x_k.$$

The $n + 1$ direction cosines X^λ of that direction in S_{n+1} which is normal at a point P of R_n to n independent (i. e., not contained in a space of less than n

* Presented to the Society (Chicago) April 22, 1905, under a slightly different title. Received for publication September 23, 1905.

dimensions) directions $ds', \dots, ds^n$ in R_n at P are found to be

$$(6) \quad X^\lambda = (-1)^{\lambda+1} \beta \begin{vmatrix} x'_1, & \dots, & x_1^{\lambda-1}, & x_1^{\lambda+1}, & \dots, & x_1^{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_n, & \dots, & x_n^{\lambda-1}, & x_n^{\lambda+1}, & \dots, & x_n^{n+1} \end{vmatrix},$$

where β denotes the reciprocal square-root of the determinant $|a_{ik}|$. It follows at once that

$$(7) \quad \sum X^2 = 1,$$

$$(8) \quad \sum Xx_k = 0 \quad (k=1, 2, \dots, n).$$

The Gaussian sphere is defined as the surface (space of n dimensions) whose coördinates are X^λ ; its equation is (7).

Let now $d\omega$ be an (n -dimensional) infinitesimal element of R_n and $d\Omega$ the corresponding element of the Gaussian sphere; then the quotient

$$(9) \quad K = \frac{d\Omega}{d\omega}$$

is KRONECKER's extension of the Gaussian curvature* — the Kronecker-Gaussian curvature of R_n .

To give its analytic expression, we define

$$(10) \quad \alpha_{ik} = \sum Xx_{ik},$$

or

$$(11) \quad \alpha_{ik} = \beta \begin{vmatrix} x'_{ik}, & x'_1, & \dots, & x'_n \\ \vdots & \vdots & \vdots & \vdots \\ x^{n+1}_{ik}, & x^{n+1}_1, & \dots, & x^{n+1}_n \end{vmatrix},$$

then

$$(12) \quad K = \frac{|\alpha_{ik}|}{|a_{ik}|}.$$

§ 2. Proof that K is expressible in terms of a_i .

We write symbolically

$$(13) \quad a_{ik} = f_i f_k = \sum x_i x_k.$$

and introduce the second covariantive derivatives, defined by

$$U^{ik} = U_{ik} - \epsilon f_{ik}(f\phi)(U\phi);$$

then

$$\sum Xx^{ik} = \sum Xx_{ik} - \epsilon f_{ik}(f\phi) \sum X(x\phi).$$

* Cf. KILLING, *Die Nicht-Euclidischen Raumformen*, p. 210.

But $\sum X(x\phi) = 0$ on account of (8), so that

$$(14) \quad \sum Xx_{ik} = \sum Xx^{ik}.$$

From (13) we deduce

$$\sum x_i x^{kl} + \sum x_k x^{il} = f_i f^{kl} + f_k f^{il} = 0$$

on account of I (84).*

Permuting in this equation i, k, l cyclically we obtain two other equations, and from these three equations we have

$$(15) \quad \sum x_i x^{kl} = 0.$$

Solving now the $n + 1$ equations

$$\sum Xx^{ik} = \alpha_{ik}, \quad \sum x_1 x^{ik} = 0, \quad \dots, \quad \sum x_n x^{ik} = 0$$

for the $n + 1$ quantities x^{ik} we obtain

$$(16) \quad x^{ik} = \alpha_{ik} X,$$

$$(17) \quad x^{ir} x^{ks} - x^{is} x^{kr} = (\alpha_{ir} \alpha_{ks} - \alpha_{is} \alpha_{kr}) X^2.$$

Therefore from (7)

$$(18) \quad \alpha_{ir} \alpha_{ks} - \alpha_{is} \alpha_{kr} = \sum (x^{ir} x^{ks} - x^{is} x^{kr}).$$

The sum in (18) can by (15) be transformed into

$$\sum x_k (x^{isr} - x^{irs})$$

which by I (111) combined with

$$\sum x_k (x\phi) = \psi_k (\psi\phi)$$

goes into

$$\epsilon(f^{isr} - f^{irs})(f\phi)\psi_k(\psi\phi).$$

But since

$$\psi_k(\psi\phi)(f\phi) = (n-1)! f_k$$

according to I (34) we have, using formula I (113), finally

$$(19) \quad \sum (x^{ir} x^{ks} - x^{rs} x^{kr}) = f^{ir} f^{ks} - f^{is} f^{kr},$$

$$(20) \quad \alpha_{ir} \alpha_{ks} - \alpha_{is} \alpha_{kr} = f^{ir} f^{ks} - f^{is} f^{kr},$$

or

$$(21) \quad \alpha_{ir} \alpha_{ks} - \alpha_{is} \alpha_{kr} = (ikrs),$$

* I quote my paper *A symbolic treatment of the theory of invariants of quadratic differential quantities of n variables*, these Transactions, vol. 4, pp. 441-469, October, 1903, by I, and my paper *Differential parameters of the first order* in this number of the Transactions by D. P.

where $(ikrs)$ is the quadruple index symbol, a quantity which involves the coefficients a_{ik} , its first and second derivatives.*

If now n is even the determinant

$$(22) \quad \Delta = |\alpha_{ik}|$$

can directly be expressed in terms of its minors of the second degree. If n is odd we apply the following general theorems on determinants which can easily be proved

$$(23) \quad \sum_{\lambda, \mu} \begin{vmatrix} \alpha_{i\lambda} & \alpha_{i\mu} \\ \alpha_{k\lambda} & \alpha_{k\mu} \end{vmatrix} \cdot \begin{vmatrix} A_{i\lambda} & A_{i\mu} \\ A_{k\lambda} & A_{k\mu} \end{vmatrix} = \Delta^2,$$

where A_{ik} denotes the minor of α_{ik} in the determinant Δ . Since every A_{ik} is again of even degree, we see that Δ^2 can be expressed in terms of the second degree minors of Δ . It follows then in general from (12) the well known † theorem that

The Kronecker-Gaussian curvature can be expressed in terms of the coefficients a_{ik} , their first and second derivatives.

I mention in passing an interesting result holding in the case where n is odd which follows at once by means of the preceding results from the general formula

$$(24) \quad \alpha_{11} \Delta^{n-1} = \begin{vmatrix} A_{22} & \cdots & A_{2n} \\ \cdot & \cdot & \cdot \\ A_{n2} & \cdots & A_{nn} \end{vmatrix}.$$

We see here that α_{11} itself can be expressed in terms of the coefficients a_{ik} . Therefore, if we call in analogy with the familiar case $n = 2$, $\sum a_{ik} du_i du_k$ the first, and $\sum \alpha_{ik} du_i du_k$ the second fundamental differential quantic, we have the theorem:

If n is odd, the coefficients of the second differential quantic are individually expressible in terms of those of the first differential quantic. The second differential quantic is determined by the first one.

§ 3. The expression of K in terms of a_{ik} .

We proceed to compute the determinant of even degree

$$(26) \quad \Delta_{\substack{i_1 i_2 \dots i_{2m} \\ k_1 k_2 \dots k_{2m}}} = \begin{vmatrix} \alpha_{i_1 k_1} & \cdots & \alpha_{i_1 k_{2m}} \\ \cdot & \cdot & \cdot \\ \alpha_{i_{2m} k_1} & \cdots & \alpha_{i_{2m} k_{2m}} \end{vmatrix}.$$

* Its unsymbolic expression is given, e. g., in BIANCHI's *Vorlesungen über Differential-Geometrie*, p. 51.

† For references on the (older) theory of the Kronecker-Gaussian curvature see KILLING, l. c., p. 263, 264.

Combining (21) with I (119) we have

$$(27) \quad \begin{vmatrix} \alpha_{i_1 k_1} & \alpha_{i_1 k_2} \\ \alpha_{i_2 k_1} & \alpha_{i_2 k_2} \end{vmatrix} = (i_1 i_2 k_1 k_2) = \epsilon f'_{i_1} f_{i_2}^2 \begin{vmatrix} (fa)'_{k_1} & (fa)'_{k_2} \\ (fa)^2_{k_1} & (fa)^2_{k_2} \end{vmatrix},$$

and therefore

$$\Delta_{\substack{i_1 i_2 \dots i_m \\ k_1 k_2 \dots k_m}} = \epsilon^m f'_{i_1} f_{i_2}^2 \dots f_{i_m}^{2m} \begin{vmatrix} (fa)'_{k_1} & \dots & (fa)'_{k_{2m}} \\ \dots & \dots & \dots \\ (fa)^{2m}_{k_1} & \dots & (fa)^{2m}_{k_{2m}} \end{vmatrix},$$

with the understanding that the $n - 1$ symbols α appearing in every successive pair $(fa)^{2\lambda-1}$ and $(fa)^{2\lambda}$ are equal, otherwise distinct.

The same determinant could have been computed by starting from any other permutation of the rows i with respective change of sign. Adding together all the expressions so obtained and dividing by $(2m)! \beta^2$ we have, by finally changing rows and columns,

$$(28) \quad \beta^2 \Delta_{\substack{i_1 \dots i_m \\ k_1 \dots k_m}} = \frac{\epsilon^m}{(2m)!} ((fa)'_{i_1} (fa)^2_{i_2} \dots (fa)^{2m}_{i_m}) (f'_{k_1} f_{k_2}^2 \dots f_{k_m}^{2m}).$$

If now n is even, we have at once the desired result:

$$(29) \quad K = \frac{1}{n! [(n-1)!]^{n/2}} ((fa)' (fa)^2 \dots (fa)^n) (f).$$

The case where n is odd presents greater difficulties. In this case we have to compute Δ^2 given by formula (23). The degree of the determinant

$$A_{i\lambda} = (-1)^{i+\lambda} \begin{vmatrix} \alpha_{11}, & \dots, & \alpha_{1, \lambda-1}, & \alpha_{1, \lambda+1}, & \dots, & \alpha_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{i-1, 1}, & \dots, & \alpha_{i-1, \lambda-1}, & \alpha_{i-1, \lambda+1}, & \dots, & \alpha_{i-1, n} \\ \alpha_{i+1, 1}, & \dots, & \alpha_{i+1, \lambda-1}, & \alpha_{i+1, \lambda+1}, & \dots, & \alpha_{i+1, n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{n1}, & \dots, & \alpha_{n, \lambda-1}, & \alpha_{n, \lambda+1}, & \dots, & \alpha_{nn} \end{vmatrix}$$

being even we can apply (28) and obtain, putting $n = 2m + 1$ and indicating each determinant by its diagonal term,

$$A_{i\lambda} = (-1)^{i+\lambda} \frac{\epsilon^m}{(2m)!} |(fa)^2_1, (fa)^3_2, \dots, (fa)^{i-1}_{i-1}, (fa)^{i+1}_{i+1}, \dots, (fa)^n_n| \\ \times |f^2_1, f^3_2, \dots, f^\lambda_{\lambda-1}, f^{\lambda+1}_{\lambda+1}, \dots, f^n_n|$$

and similar expressions for $A_{k\lambda}$, $A_{i\mu}$ and $A_{k\mu}$.

By taking according to I (119)

$$\begin{vmatrix} \alpha_{i\lambda} & \alpha_{i\mu} \\ \alpha_{k\lambda} & \alpha_{k\mu} \end{vmatrix} = (ik\lambda\mu) = \epsilon (fc)'_i (\phi c)'_k (f'_\lambda \phi'_\mu - f'_\mu \phi'_\lambda)$$

we find

$$\Delta^2 = (-1)^{i+k} \epsilon^{n+2} (fc)'_i [(fa)^2_1 \cdots (fa)^i_{i-1} (fa)^{i+1}_{i+1} \cdots] \cdot (\phi c)'_k [(\phi a)^2_1 \cdots (\phi a)^k_{k-1} (\phi a)^{k+1}_{k+1} \cdots] \\ \times \sum_{\lambda, \mu} (-1)^{\lambda+\mu} \begin{vmatrix} f'_\lambda & \phi'_\lambda \\ f'_\mu & \phi'_\mu \end{vmatrix} \cdot \begin{vmatrix} |f^2_1 f^3_2 \cdots f^\lambda_{\lambda-1} f^{\lambda+1}_{\lambda+1} \cdots|, & |\phi^2_1 \phi^3_2 \cdots \phi^\lambda_{\lambda-1} \phi^{\lambda+1}_{\lambda+1} \cdots| \\ |f^2_1 f^3_2 \cdots f^\mu_{\mu-1} f^{\mu+1}_{\mu+1} \cdots|, & |\phi^2_1 \phi^3_2 \cdots \phi^\mu_{\mu-1} \phi^{\mu+1}_{\mu+1} \cdots| \end{vmatrix}.$$

In this sum all the terms in which $\lambda = \mu$ vanish; we can, therefore, perform the summation first with respect to λ , then with respect to μ and divide the result by 2.

But since

$$\sum_{\lambda=1}^n (-1)^{\lambda+1} f'_\lambda |f^2_1 f^3_2 \cdots f^\lambda_{\lambda-1} f^{\lambda+1}_{\lambda+1} \cdots f^n_n| = |f'_1 f^2_2 \cdots f^n_n| = \frac{1}{\beta} (f)$$

and a similar reduction takes place for the other forms, the sum reduces simply to

$$\frac{1}{\beta^2} [(f'f)(\phi'\phi) - (f'\phi)(\phi'f)].$$

The expression thus obtained for Δ^2 holds for all values of i and k . Taking the sum of all these and dividing by n^2 we have

$$(30) \quad \beta^4 \Delta^2 = K^2 = \frac{\epsilon^{n+2}}{n^2} ((fc)'(fa)^2(fa)^3 \cdots (fa)^n) ((\phi c)'(\phi b)^2(\phi b)^3 \cdots (\phi b)^n) \\ \times [(f'f)(\phi'\phi) - (f'\phi)(\phi'f)].$$

Here again the $n-1$ symbols a in two consecutive terms $(fa)^{2\lambda}$ and $(fa)^{2\lambda+1}$ are equal, likewise the symbols b in $(\phi b)^{2\lambda}$ and $(\phi b)^{2\lambda+1}$.

To obtain a further reduction of the above expression we apply D. P. (1) to the product $(f'f)(\phi'\phi)$.

We have

$$(f'f)(\phi'\phi) = (f'\phi)(\phi'f) + \sum_{k=2}^n (f^k\phi)(f' \cdots f^{k-1}\phi' f^{k+1} \cdots f^n).$$

But all the terms of this sum become equal if we multiply by $((fc)'(fa)^2 \cdots (fa)^n)$. For instance in

$$(f^3\phi)(f'f^2\phi^4f^4 \cdots f^n)((fc)'(fa)^2(fa)^3 \cdots f(a)^n)$$

we can permute f^2 with f^3 and also $(fa)^2$ with $(fa)^3$ because the corresponding symbols a are equal.

In

$$(f^4\phi)(f'f^2f^3\phi'f^5 \cdots f^n)((fc)'(fa)^2(fa)^3 \cdots (fa)^n)$$

we permute f^2 with f^4 , also f^3 with f^5 and the symbols a in $(fa)^2$, $(fa)^3$ with

those in $(fa)^4$, $(fa)^5$, i. e., $(fa)^2$ with $(fa)^4$ and $(fa)^3$ with $(fa)^5$. In both cases the two expressions become equal to

$$(f^2\phi)(f'\phi'f^3\cdots f^n)((fc)'(fa)^2(fa)^3\cdots(fa)^n).$$

Thus we have the final result

$$(31) \quad K^2 = \frac{1}{n[(n-1)!]^{n+2}} ((fc)'(fa)^2(fa)^3\cdots(fa)^n)((\phi c)'(\phi b)^2(\phi b)^3\cdots(\phi a)^n) \\ \times (f'\phi'f)(f^2\phi^2\phi).$$

§ 4. *The Kronecker-Gaussian curvature as invariant of a general differential quantic.*

The $\frac{1}{2}n(n+1)$ quantities a_{ik} considered as functions of $u_1, \dots, u_n$ are not independent if $n > 2$. On account of (4) there must indeed exist $\frac{1}{2}n(n-1) - 1$ relations between them.

Let us, however, consider a differential quantic

$$(32) \quad ds^2 = \sum_{i,k=1}^n a_{ik} du_i du_k$$

where the a_{ik} are unrestricted. If now we form the quantities K (30) or K^2 (31) according as n is even or odd, then these expressions are, as is obvious immediately from their structure, differential invariants of (32).

This invariant K might properly be called the Kroneckerinvariant of the differential quantic (32). If the arc-element ds defined by (32) belongs to a space R_n contained in an euclidean space of $n+1$ dimensions, then K becomes the Kronecker-Gaussian curvature, and if $n=2$ the Gaussian curvature.

The expressions (30) and (31), especially (31), can be modified in several ways. I mention only that for $n=3$ the invariant K^2 is identical (leaving a numerical factor aside) with the invariant K_3 given in I (139).

§ 5. *The Kroneckerinvariant K of a space of λ dimensions R_λ represented as a differential parameter of a space of higher dimensions R_n containing R_λ .*

As in D. P. § 3 we define in a general space R_n of n dimensions whose arc-element is determined by

$$(33) \quad ds^2 = \sum_{r,s=1}^n a_{rs} dx_r dx_s,$$

a surface (space) R_λ of λ dimensions by the $n - \lambda$ equations

$$(34) \quad U' = \text{const.}, \dots, U^{n-\lambda} = \text{const.}$$

We shall first determine the arc-element ds of R_λ in terms of λ independent variables $u_1, \dots, u_n$ and then form the Kroneckerinvariant K of ds^2 .

For that purpose we adjoin, as in D. P. § 3, λ arbitrary functions $V', \dots, V^\lambda$ with the restriction that the functional determinant

$$(35) \quad \Delta = |V', \dots, V^\lambda, U', \dots, U^{n-\lambda}| \neq 0.$$

According to D. P. (14) the general arc-element ds in R_λ is then determined by

$$(36) \quad dx_1 = \sum_{k=1}^{\lambda} \rho^k A^{k1}, \dots, dx_n = \sum_{k=1}^{\lambda} \rho^k A^{kn},$$

where A^{kr} denotes the minor of the element V_r^k in Δ , and where $\rho', \dots, \rho^\lambda$ are λ arbitrary parameters.

On the other hand, the space R_λ defined by (34) can also be defined by expressing its coördinates x in terms of λ independent variables $u_1, \dots, u_n$. If we do this, we have for the differentials dx the expressions

$$(37) \quad dx_r = \sum_{k=1}^{\lambda} \frac{\partial x_r}{\partial u_k} du_k \quad (r=1, 2, \dots, n).$$

To make the expressions (36) and (37) equal we write first $\rho^k du_k$ instead of ρ^k in (36) so that

$$(38) \quad dx_r = \sum_{k=1}^{\lambda} \rho^k A^{kr} du_k$$

and we have now to set up the integrability-conditions of (38), i. e., the equations

$$(39) \quad \frac{\partial(\rho^i A^{ir})}{\partial u_k} = \frac{\partial(\rho^k A^{kr})}{\partial u_i}$$

following from

$$(40) \quad \frac{\partial x_r}{\partial u_k} = \rho^k A^{kr}.$$

If we denote differentiation with respect to x^k by the lower index k we find, M being any function of x ,

$$\frac{\partial M}{\partial u_k} = \sum_{s=1}^n M_s \frac{\partial x_s}{\partial u_k},$$

and by (40)

$$(41) \quad \frac{\partial M}{\partial u_k} = \rho^k \sum_{s=1}^n M_s A^{ks},$$

so that (39) becomes

$$(42) \quad \rho^k \sum_s (\rho^i A^{ir})_s A^{ks} = \rho^i \sum_s (\rho^k A^{kr})_s A^{is}$$

which reduces to

$$(43) \quad \rho^i \rho^k \sum_s (A_{,s}^{ir} A^{ks} - A_{,s}^{kr} A^{is}) = \rho^i A^{kr} \sum_s \rho_s^k A^{is} - \rho^k A^{ir} \sum_s \rho_s^i A^{ks}.$$

This equation holds for $i, k = 1, \dots, \lambda$ and $r = 1, \dots, n$.

Keeping now i and k fixed we have n equations before us. Instead of using this system of n equations, say

$$(44) \quad P_1 = 0, \dots, P_n = 0,$$

we can use the equivalent system of n equations

$$(45a) \quad \sum_{r=1}^n U_r^\alpha P_r = 0 \quad (\alpha = 1, \dots, n - \lambda),$$

$$(45b) \quad \sum_{r=1}^n V_r^k P_r = 0 \quad (k = 1, \dots, \lambda),$$

which follows from (44) and from which conversely (44) follows on account of (35).

Considering the signification of the quantities A^{kr} , we reduce equation (45a) to

$$\sum_{r,s} U_r^\alpha (A_{,s}^{ir} A^{ks} - A_{,s}^{kr} A^{is}) = 0.$$

But this equation is identically true on account of the relations which arise from differentiating the two equations

$$\sum U_r^\alpha A^{ir} = 0 \quad \text{and} \quad \sum U_r^\alpha A^{kr} = 0$$

with respect to x_s . Thus equations (45a) are satisfied without any further condition. Applying similar reductions to the equations (45b) we obtain

$$\sum_s (\rho^k \Delta_{,s} + \rho_{,s}^k \Delta) A^{is} = 0,$$

i. e.,

$$\sum_s \frac{\partial (\rho^k \Delta)}{\partial x_s} A^{is} = 0,$$

and this gives us by (41)

$$\frac{\partial \rho^k \Delta}{\partial u_i} = 0$$

for every value of i different from k . We have therefore

$$\rho^k = \frac{F'(u_k)}{\Delta},$$

and by introduction of

$$F'(u_k) du_k = du'_k$$

and writing again u instead of u' which amounts to $\rho^k = 1/\Delta$ we can state our result as follows :

If a space R_λ is given by the equations $U' = \text{const.}, \dots, U^{n-\lambda} = \text{const.}$, then by adjoining λ arbitrary functions $V'_1 \dots V^\lambda$ which have only¹ to satisfy the condition that the functional determinant

$$\Delta = |V' \dots V^\lambda U' \dots U^{n-\lambda}| \neq 0,$$

the differentials dx of R_λ can be written

$$(46) \quad dx_r = \frac{1}{\Delta} \sum_{k=1}^{\lambda} A^{kr} du_k \quad (r=1, \dots, n)$$

where A^{kr} denotes the minor of the element V^{kr} in Δ .

To find the expression for ds^2 in terms of $u_1 \dots u_n$ we apply the symbolic method, introducing symbols for the differential quantic (33) by putting

$$a_{rs} = f_r f_s.$$

We have then

$$(47) \quad ds^2 = \left(\sum_{r=1}^n f_r dx_r \right)^2.$$

To form this expression we deduce from (46), understanding by $p_1, p_2, \dots, p_n$, any n quantities

$$\sum_{r=1}^n p_r dx_r = \frac{1}{\Delta} \sum_{k=1}^{\lambda} \sum_{r=1}^n p_r A^{kr} du_k,$$

or, performing the summation with respect to r on the right side

$$\sum_{r=1}^n p_r dx_r = \frac{1}{\beta \Delta} \sum_{k=1}^{\lambda} (V' \dots V^{k-1} p V^{k+1} \dots V^\lambda U) du_k.$$

Hence

$$(48) \quad ds^2 = \frac{1}{\beta^2 \Delta^2} \sum_{i,k=1}^{\lambda} (V' \dots V^{i-1} f V^{i+1} \dots V^\lambda U) (V' \dots V^{k-1} f V^{k+1} \dots V^\lambda U) du_i du_k.$$

Let us introduce also for ds^2 as given in terms of $u_i \dots u_\lambda$ symbols by writing

$$(49) \quad ds^2 = \left(\sum_{i=1}^{\lambda} F_i du_i \right)^2;$$

then we have

$$(50) \quad F_i = \frac{1}{\beta \Delta} (V \dots V^{i-1} f V^{i+1} \dots V^\lambda U).$$

We now proceed to compute the Kronecker invariant K of the differential quantic (48) assuming λ to be an even number.

In this case we have from (29)

$$(51) \quad \lambda! [(\lambda-1)!]^{1/2} K = G = ((FA)'(FA)^2 \dots (FA)^\lambda)(F' \dots F^\lambda),$$

where the invariantive brackets $(\)$ are to be formed with respect to ds_u^2 . As

to notation we shall in all doubtful cases use the index u when reference to ds_u^2 is required.

We have from (41)

$$(52) \quad \frac{\partial M}{\partial u_i} = \frac{1}{\Delta} \sum_{r=1}^n M_r A^{ir} = \frac{1}{\beta \Delta} (V' \dots V^{i-1} M V^{i+1} \dots V^\lambda U)$$

and by D. P. (3)

$$|M' \dots M^\lambda|_u = \frac{1}{\Delta^\lambda} |M' \dots M^\lambda U' \dots U^{n-\lambda}| \cdot |V' \dots V^\lambda U' \dots U^{n-\lambda}|^{\lambda-1},$$

hence

$$|M' \dots M^\lambda|_u = \frac{1}{\Delta} |M' \dots M^\lambda U' \dots U^{n-\lambda}|,$$

and

$$(53) \quad \frac{1}{\beta_u} (M)_u = \frac{1}{\beta \Delta} (M' \dots M^\lambda U).$$

To compute β_u we apply D. P. (3) to (50) and obtain

$$\frac{1}{\beta_u} (F)_u = \frac{1}{\beta \Delta} (f' \dots f^\lambda U)$$

which leads by squaring and considering that $(F)_u^2 = \lambda!$ on account of I (17) to the required value of β_u ,

$$\beta_u^2 = \frac{\lambda! \Delta^2}{\beta^2 (f' \dots f^\lambda U)^2},$$

or, denoting the denominator which is differential parameter of ds_x^2 , by $\Delta^\lambda U$,

$$(54) \quad (f' \dots f^\lambda U)^2 = \Delta^\lambda U,$$

$$(55) \quad \beta_u = \beta \Delta \sqrt{\frac{\lambda!}{\Delta^\lambda U}}.$$

Thus we find

$$(56) \quad (M)_u = \sqrt{\frac{\lambda!}{\Delta^\lambda U}} (M' \dots M^\lambda U)$$

and

$$(57) \quad (F)_u = \sqrt{\frac{\lambda!}{\Delta^\lambda U}} (f' \dots f^\lambda U),$$

so that we have

$$((FA)'(FA)^2 \dots (FA)^\lambda) = \omega (\omega (faU)', \dots, \omega (faU)^\lambda, U),$$

where

$$\omega = \sqrt{\frac{\lambda!}{\Delta^\lambda U}}.$$

By means of D. P. (4) the right side of the above expression becomes

$$(58) \quad \omega^{\lambda+1} [(faU)', \dots, (faU)^\lambda, U] \\ + \omega^\lambda \sum_{k=1}^{\lambda} (faU)^k ((faU') \dots (faU)^{k-1}, \omega, (faU)^{k+1} \dots (faU)^\lambda, U).$$

To form G we have to multiply this by

$$(F') = \omega(f' \dots f^\lambda U).$$

I wish to show that after this multiplication has been performed each term of the sum $\sum_{k=1}^{\lambda} = T_1 + T_2 + \dots$ vanishes.

For that purpose it is sufficient to consider the first term T_1 which may be written more fully

$$T_1 = (f' a^2 \dots a^\lambda U)(f' f^2 \dots f^\lambda U)(\omega, (f^2 a^2 \dots a^\lambda U), \dots),$$

or, denoting briefly the terms $f', U' \dots U^{n-\lambda}$ by V ,

$$(59) \quad T_1 = (a^2 \dots a^\lambda, V)(f^2 \dots f^\lambda, V)(\omega, (f^2 a^2 \dots a^\lambda U), \dots).$$

By means of D. P. (1) we develop

$$(a^2 \dots a^\lambda V)(f^2 \dots f^\lambda V) = (a^2 f^3 \dots f^\lambda V)(f^2 a^3 a^4 \dots a^\lambda V) \\ + (a^3 f^3 \dots f^\lambda V)(a^2 f^2 a^4 \dots a^\lambda V) \\ + \dots \dots \dots$$

But all the terms of this sum become equal after multiplication with $(\omega, (f^2 a^2 \dots a^\lambda U), \dots)$ as one sees by permuting, e. g., in the second term the two equivalent symbols a^2 and a^3 . We have then

$$T_1 = \lambda(f^2 a^3 \dots a^\lambda V)(a^2 f^3 \dots f^\lambda V)(\omega, (f^2 a^2 \dots a^\lambda U), \dots).$$

On the other hand by permuting the equivalent symbols f^2 and a^2 in (59) we have

$$T_1 = -(f^2 a^3 \dots a^\lambda V)(a^2 f^3 \dots f^\lambda V)(\omega, (f^2 a^2 \dots a^\lambda U), \dots),$$

whence $T_1 = 0$. It follows in the same way, that also $T_2 = 0, \dots$. This leads to

$$G = \omega^{\lambda+2} ((faU)', \dots, (faU)^\lambda, U)(f' \dots f^\lambda U)$$

and to the final result

$$(60) \quad K = \frac{\lambda!^\lambda}{(\Delta^\lambda U)^{\lambda+1}} ((faU)', \dots, (faU)^\lambda, U)(f', \dots, f^\lambda U),$$

where

$$(f', \dots, f^\lambda U) = (f', \dots, f^\lambda U', \dots, U^{n-\lambda}),$$

$$\Delta^\lambda U = (f', \dots, f^\lambda, U)^2, (faU)' = (f', a', \dots, a^\lambda, U), \text{ etc.}$$

Two sets of symbols $a^2 \dots a^\lambda$ are equal in any two successive brackets $(faU)^{2k+1}$, $(faU)^{2k}$, otherwise distinct. All the symbols are symbols of the differential quantie

$$ds^2 = \sum_{i,k=1}^n a_{ik} dx_i dx_k.$$

The principles which lead to this expression of K for even values of λ will doubtless also be sufficient to solve the more complicated problem for the case of odd values of n .

Finally I wish to make an interesting application to the case $\lambda = 2$. Formula (60) gives immediately:

$$(61) \quad K = \frac{2((f\psi U), (\phi\psi U), U)(f\phi U)}{(f'\phi'U)^2 \cdot (f''\phi''U)^2}.$$

Let us first assume also $n = 2$. Then the U 's disappear and since

$$(f'\phi')^2 = (f''\phi'')^2 = 2$$

we obtain the ordinary Gaussian curvature

$$(62) \quad K = \frac{1}{2} [(f\psi)(\phi\psi)](f\phi) = \frac{1}{2(EG - F^2)} \left(2 \frac{\partial^2 F}{\partial u \partial v} - \frac{\partial^2 E}{\partial v^2} - \frac{\partial^2 G}{\partial u^2} \right) + \dots$$

Take now $n = 3$ and let R_3 be the ordinary euclidean space, i. e., $a_{ii} = 1$, $a_{ik} = 0$, if $i \neq k$. Then we have only one function U , and if we write the equation $U = \text{const.}$ in the form

$$F(x, y, z) = 0,$$

an easy computation transforms the expression (61) into the familiar form

$$(63) \quad K = - \frac{1}{\left[\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2 \right]^2} \cdot \begin{vmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} & \frac{\partial F}{\partial x} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} & \frac{\partial F}{\partial y} \\ \frac{\partial^2 F}{\partial z \partial x} & \frac{\partial^2 F}{\partial z \partial y} & \frac{\partial^2 F}{\partial z^2} & \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} & 0 \end{vmatrix},$$

so that one and the same formula (61) involves the two apparently so heterogeneous expressions (62) and (63) of the Gaussian curvature.